$m^2r^3(\pi \cos\theta + 2\sin\theta — 2\theta\cos\theta — \frac{2}{3}\sin^3\theta)$ = sum of altitudes for each base on AB. Hence $\frac{2}{3}m^4r^6(\pi\cos\theta + 2\sin\theta — 2\theta\cos\theta — \frac{2}{3}\sin^3\theta)\sin^3\theta$ = sum of triangles on AB. And

$$\int_0^{\frac{1}{2}\pi} \tfrac{4}{3}m^5r^7(\pi\sin^4\theta\cos\theta + 2\sin^5\theta — 2\theta\sin^4\theta\cos\theta — \tfrac{2}{3}\sin^7\theta)d\theta$$

$$= \tfrac{2}{3}m^5r^7\left\{\tfrac{1}{5}\pi\sin^5\theta — \frac{12}{5}\left(\tfrac{1}{5}\sin^4\theta + \frac{1.4}{3.5}\sin^2\theta + \frac{1.2.4}{1.3.5}\right)\cos\theta — \tfrac{2}{5}\theta\sin^5\theta\right.$$

$$\left. + \frac{2}{3}\left(\tfrac{1}{7}\sin^6\theta + \frac{1.6}{5.7}\sin^4\theta + \frac{1.4.6}{3.5.7}\sin^2\theta + \frac{1.2.4.6}{1.3.5.7}\right)\cos\theta\right\} + C$$

$$= \tfrac{2}{3}m^5r^7.\frac{1024}{525}. \text{ Dividing by } \tfrac{8}{3}\pi m^5r^5 \text{ gives } \frac{256r^2}{525\pi}$$

for the average area, the same as found by Mr. Seitz in his corrected result. [See *Errata*, p. 128.]

RECONSIDERATION OF SOLUTION OF PROB. 239. (*P.* 48, *VI.*)

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THERE were three solutions of this problem published, the first one furnished by me and supported by the Editor,* the second by Mr. Adcock and the third by Prof. P. E. Chase. A thorough investigation of the law of error in two dimensions, which I am making at present, has however convinced me that Prof. Chase's solution is the only correct one. In this solution, formulæ are used which Sir John Herschel developed in his Lecture on target shooting. The proof he gives, though I now admit it to be perfectly correct, failed to convince me of my error. Recently I thought of investigating the matter from a new point of view. I regarded shooting as compounded of two independent operations, viz., sighting and leveling, and I assume that errors in sighting, i. e., deviations x from the vertical y-axis and errors in leveling, i. e., deviations y from the x-axis each follow the ordinary law of error, so that if ε_x = mean error of sighting and ε_y = mean error of leveling, then

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*It will be seen, by referring to p. 50, Vol. VI, that our "support" of Mr. Kummell's solution was *conditional.* We there stated, and still assert, that the solution is correct "if the equation $y = ce^{-h^2x^2}$ represents the relation between an error and its probability."

It is obvious, however, that, in target shooting, the equation does not represent that relation; for it is apparent, from a mere statement of the case, that the center of the target is not as likely to be hit, by any single shot, as a contiguous concentric circle; and hence a very small deviation from the center, of a single shot, is not the most probable.—Editor.

$$\frac{dx}{\varepsilon_x\sqrt{(2\pi)}}e^{-(x^2\div 2\varepsilon_x^2)} = \text{probability of error } \pm x \text{ in sighting,} \qquad (1_x)$$

$$\frac{dy}{\varepsilon_y\sqrt{(2\pi)}}e^{-(y^2\div 2\varepsilon_y^2)} = \text{“ “ “ } \pm y \text{ in leveling,} \qquad (1_y)$$

consequently $\frac{dx\,dy}{2\varepsilon_x\,\varepsilon_y\pi}e^{-(x^2\div 2\varepsilon_x^2)-(y^2\div 2\varepsilon_y^2)}$ = probability of hitting the point (x, y). (2)

I shall not consider here the general case any further, but assume, which is practically sufficient, $\varepsilon_x = \varepsilon_y = \varepsilon$. Transforming (2) to polar coordinates by assuming $x = r\cos a$ and $y = r\sin a$, when $dxdy$ must be replaced by $r\,dr\,da$, we have

$$\frac{r\,dr\,da}{2\varepsilon^2\pi}e^{-\frac{r^2}{2\varepsilon^2}} = \text{probability of hitting the point } (r, a), \qquad (3)$$

and
$$\frac{r\,dr}{\varepsilon^2}e^{-\frac{r^2}{2\varepsilon^2}} = \text{prob. of shooting a dist. } r \text{ from center.} \qquad (4)$$

This probability has a maximum, viz., if

$$0 = \frac{1}{\varepsilon^2} - \frac{r^2}{\varepsilon^4},\ \therefore\ r = \varepsilon, \qquad (5)$$

which means, that if we divide a target record in (infinitesimal) rings of eq'l width, then the one, whose radius $= \varepsilon$, will contain the greatest number of hits. This is then the most probable shot.

Integrating (4) from the center to the distance r, we have, if n = total number of shots and n_r = the number of hits on circle, radius r,

$$\frac{n_r}{n} = 1 - e^{-\frac{r^2}{2\varepsilon^2}};\ \therefore\ \varepsilon = \frac{r}{\sqrt{2l[n\div(n-n_r)]}} \qquad (6)$$

We have then $n_\varepsilon \div n = 1 - e^{-\frac{1}{2}}$, or $n_\varepsilon = 0{,}395\,..\,n = \frac{2}{5}n$ nearly. (7)

If then we count $\frac{2}{5}n$ shots nearest the center, then the farthest of these will be the most probable shot nearly; and it is obvious that this is the most accurate value of it we can expect to obtain by simply counting shots, because shots at this distance will be nearer together when revolved on the same line than anywhere within or without.

Herschel employs another quantity for comparing skill in shooting, viz., the radius of a circle which should receive half the number of shots. This circle I call the even chance circle, and if ρ denotes its radius we have in (6)

$$\tfrac{1}{2} = e^{-(\rho^2\div 2\varepsilon^2)};\ \therefore\ \rho = \varepsilon\sqrt{(2l2)} = 1.177..\varepsilon \qquad (8)$$

Eliminating ε from (6) and (8) we have

$$\left(\frac{1}{2}\right)^{\rho^2} = \left(\frac{n-n_r}{n}\right)^{r^2}; \qquad (9)$$

$$\therefore \rho = r\sqrt{\left(\frac{l\dfrac{n-n_r}{n}}{l\frac{1}{2}}\right)} = r\sqrt{\left(\frac{l\dfrac{n}{n-n_r}}{l\,2}\right)}. \tag{10}$$

This agrees with Herschel's formulæ which Prof. Chase employs.

I abstain from developing the more accurate formulæ for determining these constants from the sum of the squares of the distances, or from the sum of the distances (the string), also in the case of stray shots, and give in conclusion the most probable shots of the marksmen A and B. They are,

For marksman A, $\varepsilon_a = \dfrac{5}{\sqrt{(2l\frac{100}{64})}} = 5.292.$

For marksman B, $\varepsilon_b = \dfrac{10}{3\sqrt{(2l\frac{100}{36})}} = 4.664.$

PROBLEM 436.—"Integrate the equation

$$x^m y^n(aydx + bxdy) = x^{m'}y^{n'}(a'ydx + b'xdy)."$$

SOLUTION BY PROF. W. W. BEMAN.—Evidently $x^{-m-1}y^{-n-1}$ is an integrating factor of the first member, whence an integral of $x^m y^n(aydx+bxdy) = 0$, is $\log(x^a y^b) = c$, or $x^a y^b = C$.

The general form of integrating factor of the first member is, then,

$$x^{-m-1}y^{-n-1}\,\varphi(x^a y^b).$$

In the same way we may obtain a general integrating factor for the second member, $\quad x^{-m'-1}y^{-n'-1}\,\psi(x^a y^b).$

That these two may be equal, we must have

$$x^{m'+1}y^{n'+1}\,\varphi(x^a y^b) = x^{m+1}\,y^{n+1}\,\psi(x^a y^b).$$

Let $\varphi(x^a y^b) = (x^a y^b)^r$, $\psi(x^a y^b) = (x^a y^b)^s$, r and s being indeterminate. Then

$$x^{m'+1}\,y^{n'+1}\,(x^a y^b)^r = x^{m+1}\,y^{n+1}\,(x^a y^b)^s.$$

$$\therefore\ m'+ar = m+a's, \quad n'+br = n+b's;$$

$$r = \frac{a'(n-n')-b'(m-m')}{a'b-ab'}, \quad s = \frac{a(n-n')-b(m-m')}{a'b-ab'}.$$

The integrating factor becomes

$$x^{ar-m-1}\,y^{br-n-1} = x^{a's-m'-1}\,y^{b's-n-1};$$

$$\therefore\ x^{ar-1}\,y^{br-1}(aydx+bxdy) = x^{a's-1}\,y^{b's-1}(a'ydx+b'xdy).$$

Integrating, $\quad \dfrac{1}{r}x^{ar}\,y^{br} = \dfrac{1}{s}x^{a's}\,y^{b's} + C.$

When $a'b = ab'$ the equation is immediately integrable.

This solution is based upon that of a special form of the equation given, found in Hoüel's Calcul Infinitesimal.